FOOTNOTES ON THE FOUNDATIONS OF GAME THEORY

F1. UNIVERSAL SOLVERS

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This is the first paper in *Footnotes on the Foundations of Game Theory*. It provides verifiable information of a universal solver **g[]**, for finite pure-strategy Nash Equilibria, without disclosing code. In doing so, theory at work is displayed. In particular, this feat exploits a bridge between static games and Replicator dynamics to show that only predicted symmetric NE are stable fixed points. A theoretical discussion from the perspective of a Machiavellian ruler accompanies this mathematical treatment.

The second part establishes a link between the number of solutions and their character in terms of symmetric (diagonal) solutions and coupled (off-diagonal) in symmetric matrices with heuristics. Data generated with **g[]** is used to test hypotheses. This method is improved on with a statistical model to predict the distribution of the number of solutions. Data supports the model. All results follow from simple or common-knowledge concepts and definitions. More general solvers will be discussed in future work.

Keywords:

Game Theory, *Applied Mathematics*, *Social Science*, *Evolution*.

Please note: *This paper is a first draft, revision may result in major updates*. Comments, requests or questions are welcome.

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UNIVERSAL SOLVERS

This text belongs in the series *Footnotes on the Foundations of Game Theory*. It introduces the universal solver **g[]** for pure-strategy Nash Equilibria in static finite games[1]. This presentation establishes the feasibility of the project, and provides glimpses of its scientific potential, beyond the ability of solving thousands of games instantly. In particular, it is feasible to make and test predictions about a large number of games or huge ones. Ultimately, this paper embarks on a route to explore the set of all possible games, in turn the basis for a wide range of thought experiments and theorems within the Sciences, Mathematics and Philosophy. Theory is discussed alongside the solver, from the perspective of a Machiavellian ruler.

It is known mathematicians are able to provide proofs without revealing much about of the underpinnings of their work. Likewise, this paper does not print code but provides information to check the following constraints:

(a) Proposed solutions indeed are NE.
(b) All other outcomes are unstable.
(c) Games are not rigged in order to falsely emulate a solver.
(d) Provide data to check all of the above.

The reader will also be introduced to fundamental connections between seemingly disparate fields. In particular, this paper explores the connection between static and dynamic games, with implications on a much heated debate. Satisfying a-d amounts to problem solving, which displays theory at work in the process.

This Footnote starts by generating of data from known distributions, and then proceeds with automated solving by virtue of **g[]**. The cost of deception (c), is high and increasing because manipulations nevertheless must (i) be such

[1] Coded by me at the outset this side-project, starting April 2022. I want students to learn and create with the principles of social science by constructing, and using, universal solvers.

that all *pure-strategy* NE are found (ii) conform to a well-known distribution with specified parameters. In addition (iii), big enough games will be computed in sizeable quantities. This combination will work as a deterrent burdening deception in proportion to the number of games solved with transparency. Future *Footnotes on the Foundations of Game Theory* will expand on universal mixed-strategy solvers, dynamic games, Markov processes and Evolutionary Game Theory. In addition, predicting NE outcomes from a particular distribution of the incentive structure has theoretical appeal. This paper takes important steps in that direction, and will be expanded on in updates and future footnotes.

To check large games is cumbersome and arguably requires a similar device in the hands of the readers. In order to address this while respecting a-d, Evolutionary Game Theory will be used to provide means to evaluate results. Articulation of games in terms of *Replicator Dynamics* bridges symmetric NE and fixed points, where myopic players adapt to the average (field) play. In this manner I effectively provide data and well-known equations to check the solution, without disclosing crucial information about **g[]** etc. I will demand some goodwill and/or effort from the readers when evaluating a batch of 50, seven-player randomly generated games and their solutions. The sheer size should be enough to deter manipulation and/or manual search. The Replicator Dynamics approach is introduced with a (50x50) 2-player game in order to avoid more theorems or analysis resulting from n>2 *players*. In this setting, the *strategies* can be interpreted as *types* of a *population*.

Elusive topics such as the stark connection between method, theory and ruminations on society are given systematic treatment. In similar fashion, informal reasoning on the correspondence between evolution, myopic behaviour and rational behaviour in society is given sound theoretical basis.

DATA g [ ]

50 games with seven players are solved. The number of strategies for each player is drawn from a *Uniform Distribution*[1,3]. The incentive structure is generated with a *Poisson* of mean 3 for each player strategy; and at each contingency of the game. All relevant data is uploaded with description.

RESULTS 1

Strategy indexed are reserved for each player with indexes: ABC, DEF,… STU. Table 1 provides all the states & NE for the first of the 50 games.

T1. NE Game 1/50

STATE	P1	P2	P3	P4	P5	P6	P7	STATE	P1	P2	P3	P4	P5	P6	P7
ADGJMPS	4	1	2	5	5	2	4	CDGJOPS	2	2	3	3	2	1	2
ADGJNPS	5	3	1	4	9	4	7	CDGKMPS	5	2	4	3	1	2	2
ADGJOPS	2	2	4	4	2	4	6	CDGKNPS	3	3	5	2	3	4	1
ADGKMPS	6	2	2	4	2	3	0	CDGKOPS	5	4	5	5	7	1	1
ADGKNPS	6	3	3	3	2	2	6	CDGLMPS	4	4	4	3	7	2	2
ADGKOPS	3	2	2	2	6	4	1	CDGLNPS	2	3	2	2	0	5	2
ADGLMPS	3	1	2	1	2	5	3	CDGLOPS	3	3	6	1	5	1	4
ADGLNPS	7	2	0	3	2	6	1	CDHJMPS	2	2	1	3	3	7	4
ADGLOPS	3	7	0	2	5	3	6	CDHJNPS	1	2	1	2	4	1	3
ADHJMPS	4	3	3	3	1	2	2	CDHJOPS	1	3	3	2	2	3	3
ADHJNPS	2	1	2	3	5	5	4	CDHKMPS	1	2	4	2	2	1	4
ADHJOPS	4	2	4	4	3	5	1	CDHKNPS	2	7	3	2	8	2	1
ADHKMPS	6	3	2	5	5	2	3	CDHKOPS	1	10	4	1	5	6	3
ADHKNPS	5	4	1	1	4	2	3	CDHLMPS	3	4	3	6	3	1	1
ADHKOPS	4	4	1	4	1	4	2	CDHLNPS	3	5	2	2	3	1	2
ADHLMPS	7	3	3	1	1	4	2	CDHLOPS	4	3	5	3	4	4	2
ADHLNPS	4	1	2	5	1	3	2	CEGJMPS	4	4	2	1	5	5	3
ADHLOPS	2	5	0	1	1	7	4	CEGJNPS	2	1	4	2	3	3	8
AEGJMPS	3	6	5	0	2	3	6	CEGJOPS	1	1	1	1	2	3	4
AEGJNPS	2	4	3	1	2	4	4	CEGKMPS	2	4	3	7	4	2	2
AEGJOPS	5	3	1	6	2	0	3	CEGKNPS	2	4	1	1	0	4	3
AEGKMPS	5	1	3	6	2	4	1	CEGKOPS	1	2	1	3	1	4	4
AEGKNPS	4	4	2	2	4	2	3	CEGLMPS	3	4	1	2	4	3	4
AEGKOPS	7	4	3	1	0	2	2	CEGLNPS	5	3	2	5	4	4	5
AEGLMPS	6	2	6	2	3	4	0	CEGLOPS	4	5	3	2	2	2	3
AEGLNPS	3	5	1	3	2	2	3	CEHJMPS	1	3	1	3	3	2	1
AEGLOPS	2	4	1	4	2	3	4	CEHJNPS	3	4	3	5	3	4	3
AEHJMPS	2	3	5	0	2	3	1	CEHJOPS	4	4	3	3	2	6	3
AEHJNPS	9	3	0	3	5	0	3	CEHKMPS	2	2	3	3	1	1	2
AEHJOPS	5	4	4	3	0	5	3	CEHKNPS	2	5	3	2	4	3	0
AEHKMPS	4	3	3	6	0	1	4	CEHKOPS	3	1	2	2	0	3	7
AEHKNPS	4	1	5	6	3	3	6	CEHLMPS	2	4	4	1	6	5	3
AEHKOPS	2	3	2	4	7	3	1	CEHLNPS	3	3	1	3	6	5	9
AEHLMPS	1	1	2	2	4	5	0	CEHLOPS	5	3	3	3	3	4	4
AEHLNPS	4	1	4	3	3	5	3	**NE**	ADHKMPS; CDGLMPS; CEGLNPS; CDGKOPS						
AEHLOPS	4	0	2	3	3	4	5	**Strat**	{{A,B,C},{D,E},{G,H},{J,K,L},{M,N,O},{P},{S}}						
BDGJMPS	3	2	5	2	4	3	4								
BDGJNPS	2	4	2	3	2	4	1								
BDGJOPS	4	2	1	3	0	1	2								
BDGKMPS	2	8	3	1	2	0	4								
BDGKNPS	2	4	3	3	2	0	2								
BDGKOPS	2	2	3	3	3	3	0								
BDGLMPS	4	2	3	4	4	4	6								
BDGLNPS	5	8	6	2	6	5	6								
BDGLOPS	2	2	8	4	3	5	5								
BDHJMPS	1	1	2	3	4	3	5								
BDHJNPS	3	1	1	4	1	4	5								

BDHJOPS	2	5	3	3	3	3	2
BDHKMPS	3	5	4	4	5	1	4
BDHKNPS	5	4	2	0	3	4	2
BDHKOPS	2	4	2	3	2	4	4
BDHLMPS	3	2	1	2	3	4	3
BDHLNPS	6	3	5	2	2	2	1
BDHLOPS	2	4	2	8	3	2	5
BEGJMPS	2	3	4	7	1	3	2
BEGJNPS	5	2	4	3	4	3	3
BEGJOPS	4	4	2	5	3	1	3
BEGKMPS	4	2	1	1	1	6	4
BEGKNPS	2	5	3	1	3	2	2
BEGKOPS	4	2	4	5	5	1	1
BEGLMPS	6	2	5	1	3	2	3
BEGLNPS	2	5	2	0	1	1	6
BEGLOPS	4	1	1	7	3	4	6
BEHJMPS	1	3	1	2	0	4	3
BEHJNPS	3	3	3	2	1	7	3
BEHJOPS	4	1	3	2	4	2	4
BEHKMPS	2	2	3	1	2	3	5
BEHKNPS	3	6	4	1	4	4	1
BEHKOPS	1	8	3	4	5	5	2
BEHLMPS	6	6	2	5	1	3	4
BEHLNPS	3	6	3	5	1	4	2
BEHLOPS	3	1	1	2	3	3	3
CDGJMPS	2	3	4	1	3	5	3
CDGJNPS	2	3	2	3	3	3	4

The strategy set is {{A,B,C},{D,E},{G,H},{J,K,L},{M,N,O},{P},{S}}. The solutions for the 50 games are:

$$\Big\{\begin{pmatrix}\mathrm{ADHKMPS}\\\mathrm{CDGLMPS}\\\mathrm{CEGLNPS}\\\mathrm{CDGKOPS}\end{pmatrix}, \begin{pmatrix}\mathrm{CEGJNPS}\\\mathrm{BDIJMQT}\\\mathrm{BEHJMRT}\end{pmatrix}, (\mathrm{AFGKMPT}), \{\}, \begin{pmatrix}\mathrm{BEHKMPS}\\\mathrm{ADGJMPT}\end{pmatrix}, \begin{pmatrix}\mathrm{BEGKOPS}\\\mathrm{AEGJOQS}\\\mathrm{BEGKMPT}\end{pmatrix}, (\mathrm{BDGJMPS}), \begin{pmatrix}\mathrm{ADGLNPS}\\\mathrm{BDGJOPS}\\\mathrm{ADHKOPS}\\\mathrm{ADGKOQS}\\\mathrm{ADGJMRS}\\\mathrm{ADHLMRS}\end{pmatrix},$$

$$\begin{pmatrix}\mathrm{CDGJMPS}\\\mathrm{ADHJMPS}\\\mathrm{ADGKOPS}\end{pmatrix}, (\mathrm{BDHJOPS}), \begin{pmatrix}\mathrm{BEGKMPS}\\\mathrm{BEIJNPT}\\\mathrm{AFGLNPT}\\\mathrm{BDGLMQT}\\\mathrm{AFGKNRT}\\\mathrm{ADHKMRU}\end{pmatrix}, (\mathrm{AEHJOPS}), \begin{pmatrix}\mathrm{ADGJNPS}\\\mathrm{BDGJOPS}\\\mathrm{BDGJNQS}\end{pmatrix}, (\mathrm{BEGKMPS}), \begin{pmatrix}\mathrm{CDIJOPS}\\\mathrm{CEHJOPU}\end{pmatrix},$$

$$\{\}, (\mathrm{AEIKMRS}), \begin{pmatrix}\mathrm{BEGKOPS}\\\mathrm{ADGKMRS}\\\mathrm{CEGJNRS}\\\mathrm{BDGLNRS}\end{pmatrix}, \begin{pmatrix}\mathrm{BEGJMPS}\\\mathrm{BDGJMQT}\\\mathrm{CEGJMQT}\\\mathrm{CDGJMQU}\\\mathrm{BEGJMQU}\end{pmatrix}, \{\}, \begin{pmatrix}\mathrm{ADGJMPS}\\\mathrm{ADIJMPS}\end{pmatrix}, \begin{pmatrix}\mathrm{ADGJMPS}\\\mathrm{AEHKNPS}\\\mathrm{ADGKMPT}\\\mathrm{AFHJMQT}\\\mathrm{AFHKNQT}\end{pmatrix}, (\mathrm{BEGJMPS}),$$

$$\begin{pmatrix}\mathrm{ADGJNPS}\\\mathrm{ADHKNPS}\\\mathrm{CDGLMQS}\\\mathrm{AEHJNRS}\end{pmatrix}, \begin{pmatrix}\mathrm{BEGJMPT}\\\mathrm{BEHJMPT}\end{pmatrix}, (\mathrm{BDGJNPS}), \begin{pmatrix}\mathrm{ADGKMPS}\\\mathrm{CDGJNPS}\\\mathrm{BDGKNPT}\end{pmatrix}, (\mathrm{AFGKMPS}), \begin{pmatrix}\mathrm{ADGKMPS}\\\mathrm{ADGKMPT}\end{pmatrix}, \begin{pmatrix}\mathrm{ADHJNPT}\\\mathrm{BDGKNPT}\end{pmatrix},$$

$$\begin{pmatrix}\mathrm{CFHKNPS}\\\mathrm{BEIJNPT}\\\mathrm{ADIKNQT}\end{pmatrix}, \begin{pmatrix}\mathrm{AFHLMPS}\\\mathrm{AFHJOPS}\end{pmatrix}, \begin{pmatrix}\mathrm{BDHJMQS}\\\mathrm{BDHJMPT}\\\mathrm{BDGJMQT}\\\mathrm{BDHJNQT}\end{pmatrix}, \begin{pmatrix}\mathrm{BFGKMPS}\\\mathrm{AFGJNPS}\\\mathrm{ADIKNPS}\\\mathrm{AEIJMPT}\\\mathrm{BDHKNPU}\end{pmatrix}, (\mathrm{ADIJMPS}), \begin{pmatrix}\mathrm{ADHJNPS}\\\mathrm{AFHJMQS}\\\mathrm{ADGJNQU}\end{pmatrix}, \{\},$$

$$\begin{pmatrix}\mathrm{AEHKOPT}\\\mathrm{BFHKOPT}\end{pmatrix}, \begin{pmatrix}\mathrm{BDGJNPS}\\\mathrm{BDGJOPS}\end{pmatrix}, \{\}, \begin{pmatrix}\mathrm{BEGJMQS}\\\mathrm{BDIJNQU}\end{pmatrix}, \begin{pmatrix}\mathrm{AEIJMPT}\\\mathrm{AFHKMPT}\end{pmatrix}, \begin{pmatrix}\mathrm{AEGJNPS}\\\mathrm{ADGKNPT}\\\mathrm{AEGKNPT}\end{pmatrix}, (\mathrm{BEGKMPS}),$$

$$(\mathrm{BDGKMQS}), (\mathrm{BDGKMPS}), \begin{pmatrix}\mathrm{BEHJOQS}\\\mathrm{AEGJMRS}\\\mathrm{AEIJORS}\\\mathrm{CEIKORS}\end{pmatrix}, \begin{pmatrix}\mathrm{AEGJMQS}\\\mathrm{AEGJMPU}\\\mathrm{CFGJNPU}\\\mathrm{AEGLNQU}\end{pmatrix}, \begin{pmatrix}\mathrm{AEGJMPS}\\\mathrm{BFGJMPS}\\\mathrm{BDGKMPS}\\\mathrm{AEGKMQS}\end{pmatrix}, \begin{pmatrix}\mathrm{ADHJMPS}\\\mathrm{ADGJMQS}\\\mathrm{ADGJMQT}\\\mathrm{ADHJMQT}\\\mathrm{ADGJMQU}\end{pmatrix}\Big\}$$

Full dataset is provided online. Note some sets are empty, which means at least a mixed-strategy equilibrium. Recall *games are restricted to pure-strategy play by assumption*. Such games are however common in the literature with countless applications. A Footnote on a mixed-strategy solver will accompany these findings in the future.

CONCLUDING REMARKS 1

g[] deals with finite pure-strategy games. The solutions of 50 games were computed instantly with obsolete hardware. Thus, students or researchers with limited resources also benefit from this software. The statistical approach taken so far does not only safeguard code, but provides glimpses of future venues for research. Regularities such as the number of equilibria and their properties, given distributions underpinning incentives, is an arena for statistical inference and analysis. Huge games can be predicted or explored in this manner. Such feats are relevant in applications for purpose of institutional design, theory creation, and hypothesis testing. Statistical models and predictions about how many equilibria are to be expected given distributions underpinning incentives are treated in *Statistical Properties of NE*.

EVOLUTIONARY GAME THEORY g[]

More realistic assumptions concerning the behavioural dispositions of isolated individuals and groups have gained traction over the years. One of the reasons is quite fierce critique of perfect rationality, especially unrealistic calculation prowess, presumably envisaged by Nash or Von Neuman. However, the duality between equilibria reached by perfectly rational individuals on one hand; and myopic ones through trial and error on the other, was emphasised from the outset. Such remarks remain relevant to understand one of the more powerful aspects of Game Theory. As I have argued elsewhere, multiplicity of equilibria can be used for purposes of institutional design and, among other things, be conceptualised as expressions of intent from a rational planer.

Moreover, critique of systems and outdated theory in their defence is not aided by discarding this link by means of hand waving. This connection provides clues to why systems prevail, in particular why flawed apologetic theories persist. In this setting myopic individuals converge to an important set of NE, guided by trial and error. Furthermore, more realistic assumptions about the psychology of individuals and non-pecuniary motives may have negligible impact on patterns of interaction and final outcomes, when not compatible with overarching economic structures. Nevertheless, it is easy to introduce them when employing universal solvers.

Replicator dynamics starts with a population consisting of n different types. The shares of the population playing (being) one of these types is denoted x_i, and expressed in the vector $\mathbf{x}$ containing the population distribution of these shares. The evolution of $\mathbf{x}$ is given by the following n-1 differential equations *at any point of time*

$$x'_i = x_i\mathbf{[}\pi(s_i,\mathbf{x}) - \underline{\pi}(\mathbf{x})\mathbf{]}$$

These state that the share of the population using a particular strategy/being a type is determined by the difference between the payoff of such strategy $\pi(s_i,\mathbf{x})$ and the average payoff in the population $\underline{\pi}(\mathbf{x})$. The former payoff simply is the expected value of s_i given the population distribution $\mathbf{x}$. The latter is computed from the former by taking the expected value of $\pi(s_i,\mathbf{x})$ instead. By convention, time (t) is suppressed in notation.

This two-player setting can be interpreted as follows: a pure strategy s_i of player one gives rise to a payoff (π) which reflects how it fares against nature. The latter plays some type with certain probability reflecting the population distribution. Therefore, symmetric matrices are readily consistent with this setup. Note however, that types can be interpreted as subsets of one entity.

The evolutionary game is constructed as follows. A symmetric nxn matrix (n=50) is generated with a **Poisson[3]** distribution *until a symmetric solution emerges*. Payoffs when two of the same type meet are the same for both. A game of this size is costly to generate with correct distribution and solutions, without a device such as **g[]**. This holds true even with a corresponding system of replicator-dynamics differential equations. The existence of symmetric NE must still be confirmed with such roundabout way, which in effect becomes taxing constraint in terms of time and computation.

Symmetric pure-strategy NE are fixed point in the corresponding Replicator-Dynamics systems for n>2. This setup and result is common-knowledge within Evolutionary Game Theory. Hence, the reader is encouraged to consult the mathematical research literature on this topic if needed.

RESULTS 2

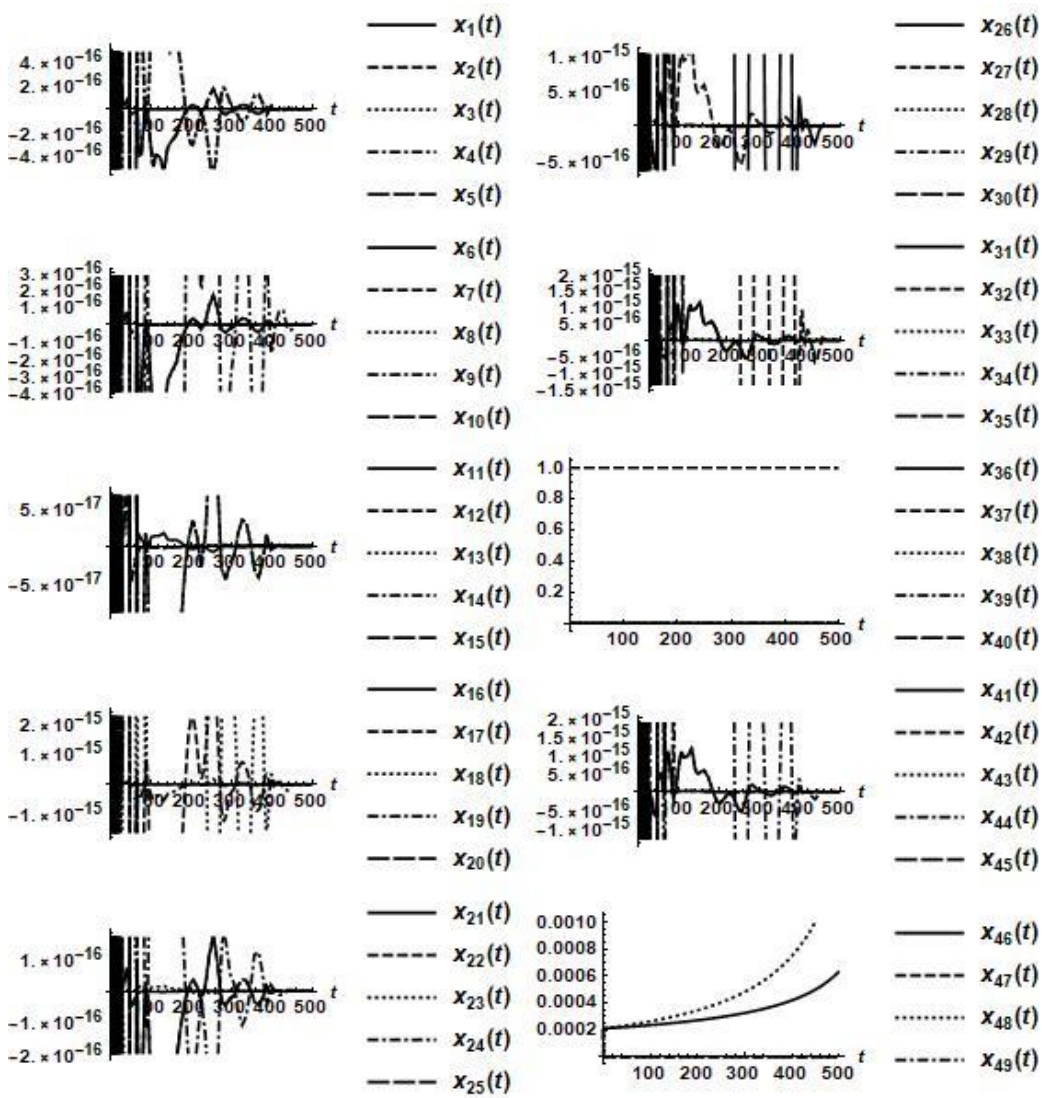
Data for this experiment is provided at online. There are three pure-strategy NE. One chooses 22 and the other 29; or both choose strategy 37.

In the first experiment, the latter symmetric NE is given a share of $x_{37}(0) = 99\%$. The rest are given the uniform distribution[2]. As figure 1 shows, there is immediate convergence and stability throughout $t \in [0,500]$. This system would have reached steady state fast if simulation ruled out negative values, i.e. in any conceivable (non-subjective) realistic setting interpreting x_i as shares in $[0,1]$. But the results are nevertheless strengthened as such detours may induce drifts, which can make the NE drop to zero, if the mean becomes

[2] The uniform distribution is adapted to the initial condition of the NE type. Each type's initial condition is random e.g. $x_i \sim U[0,1/a]$, and a must be s.t. $\mathbf{x}$ satisfies a certain Kolmogorov condition.

negative. Otherwise, once a share reaches zero it stays there. In effect, the absence of a **[**0, 1**]** restriction makes the stability test more demanding.

F1. Myopic Evolution & Symmetric NE (X_{37})



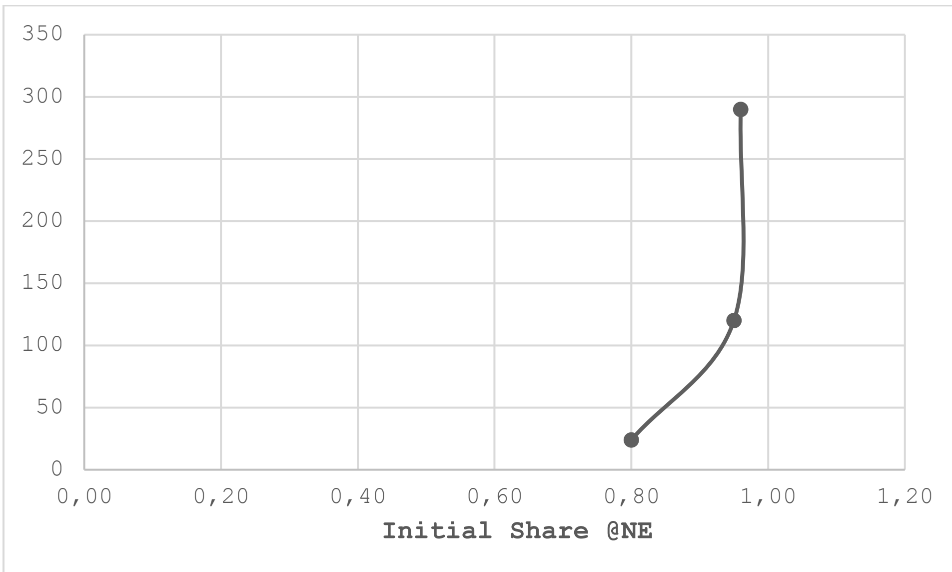
The state vector **x** evolves to a homogenous population with a single type, namely x_{37} which is the symmetric pure-strategy Nash Equilibrium of the corresponding 50x50 *static game*.

The second experiment searches for a lower bound for stability. At $x_{37}(0) \approx$ 95 %, and the rest uniformly distributed, the equilibrium holds until t=120.

At $x_{37}(0) \approx 4/5$ there is an initial convergence, followed by a sharp decline towards zero at $t \in$ **[**24, 29**]**.

Instead it is x_{36} which stabilises between **[**1/5, 2/5**]**, suggesting a polymorphic equilibrium, which may correspond to a mixed-strategy NE. Analysis of such cases is postponed until the mixed-strategy solver is presented.

F2. Stability Period & Initial Conditions



Clearly, steady-state time increases dramatically about a 95 % threshold.

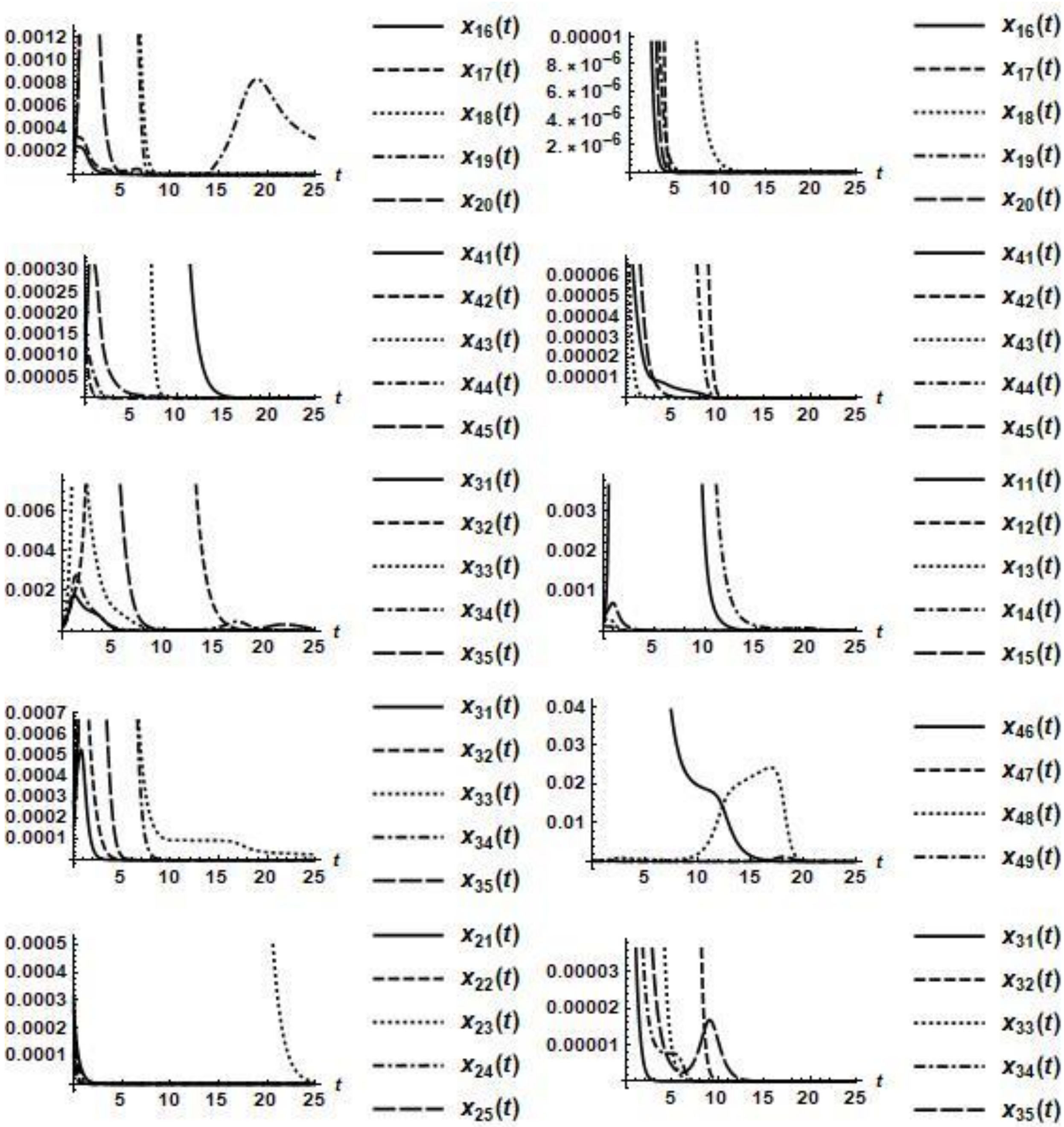
The third experiment takes a random sample of size 10 from the set of types and checks convergence with initial conditions $x_i(0) \approx 99\%$, and the rest uniformly distributed as above. The favoured types are

$$\{18, 43, 35, 33, 23, 17, 42, 14, 46, 32\}$$

None of the types in the sample, which consists of 1/5 of the total, are stable. Oddly enough, x_{37} increases and approaches 1 when x_{17} is favoured. Table 3 below shows these in groups.

Counting begins at the first column and proceeds downwards, i.e. 18 is in column 1, row 1; and 17 in column 2, row 1. No stable NE-convergence detected at t = 500.

F3. 10 Favoured & Stability



The reader is encouraged to consider a more analytical route, e.g. linearisation, or simulate the full sample of alternatives to the symmetric NE.

CONCLUDING REMARKS 2

This section utilised a bridge between Evolutionary Game Theory and symmetric NE. This take also highlights a link between dynamics with myopic agents on one hand; and static one-shot games, with equilibria sustained by perfectly rational players, on the other. In doing so, I have provided verifiable information of a universal solver without disclosing code.

Replicator Dynamics confirmed the stability of a symmetric pure-strategy equilibrium, and the instability of others. Allowing negative state-variables of type distributions is unrealistic but makes the result more robust. In essence this relaxation of restrictions works as noise which perturb equilibria. Notwithstanding, there is a sharp exponential increase in steady-state time when the symmetric NE is given a share above 95%. A methodological point is made. Useful inferences about application of a class of models may be drawn from a particular conceptually flawed one, effectively unsuitable for realistic applications, at a fundamental mathematical level. Of course, it is not unthinkable idiosyncratic beliefs assign negative probabilities or shares.

MACHIAVELLIAN REMARKS

Stability follows from symmetric pure-strategy NE. Such outcomes may in particular be desired by a ruler, who will be the protagonist in what follows. A ruler can benefit from devices ensuring specific types are favoured, in order to advance such aims. Types may correspond to individuals, such as the so-called masses; certain ideas; social outcomes; institutions or other wants at stake. All of these will be referred to as *desires*.

In mathematical terms, a wise ruler will be interested in the diagonal of the incentive matrix. The probability distribution should be such that benefits are relatively favourable when two of the same desired kind meet, as to ensure that they are the best response to each other. Thus whatever game or scheme the ruler is using, should result in a probability distribution favouring the diagonal elements as above. The good news for the rulers is that there are plenty of opportunities to achieve this. If discrimination among different types is allowed, then manipulation of incentives of one type on the diagonal will not, by mathematical necessity, affect the incentives to include or exclude others in a wide array of settings. This is once again true by virtue of definition of what a diagonal element is.

The wise ruler realises diagonal elements ensure stability but are only *n* in a nxn matrix. A 1/n rule of thumb suggest stability is an increasingly unlikely proposition with a growing number of types. But even a ruler cannot force distributions by decree if these are ruled out by higher laws. Whatever random process is used to assign an equilibria, only the first entry of a tensor will leave it intact for sure. To avoid inconsistency, a naïve assignment rule would not remain unaltered in general. For instance, if a diagonal gets a NE, then such may be an outlier with high scores for both players, and thus exclude a number of possibilities for the rest.

Consider a *symmetric matrix*. It reflects if a certain payoff is given when a certain type meets a certain other, then the same is given to corresponding type whomever 'player plays'. True that if tensor m_{ij} is an off-diagonal NE, then tensor m_{ji} typically also is one in this setting. Such off-diagonal symmetry is called *coupled* below. However, given a diagonal NE, such would only rule out a $1/n^2 \sim 0$ of the possible symmetric NE. But it would at most rule out a $(2n-1)/n^2 = 2/n - 1/n^2 \sim 2/n$ share off-diagonal. An off-diagonal NE could rule $2(2n-2)$ possible NE, i.e. a share $\sim 4/n$, but only $2/n^2 \sim 0$ on diagonal. This will work to push the frequency of NE upwards over 1/n of the solutions.

A ruler may by providence allow individuals to choose who they want to be at an individual level, from now on called freedom, as such generosity is endowed with increasing returns. The more flexible individuals are in this regard the better, as long as they are prepared to adapt to the realities of the games. In such manner, the whole population N would at most be at the disposal of the ruler. In the lingo of the ruler, individuals with this ability are said to be *freer*, even if the opening discussion has made clear that such decisions may not be conscious or rational.

Freer individuals will be able to fulfil a greater number of combinations of desires for the ruler. One way of combining freer individuals with desired stable outcomes is to bolster sentiment for those within the same type, i.e. to induce homophile incentives. Either benefits of homophile interaction increase or the benefits for interaction across types decrease. Therefore, the wise ruler induces some to consort with more ease, while upsetting others, and then they will always be faithful to ultimate desires.

Symmetry emerges due to its desirable property to attain a preferred distribution of types by means of incentives, when favoured by rulers. As a conjecture a consequence of periodic myopic design by favoured types, in resulting repeated games favouring those types, hence reproduced.

However, although a multitude of types can be preserved as NE in static games, increasing incentives by means of a favourable distribution, Replicator Dynamics typically converge to homogeneity in pure-strategy settings. If the reader thinks a special case of equal and much higher payoffs for diagonals will do, then you are in good company, but it is wrong. Experiments analogous to the aforementioned show differently also in this case.

To the extent these limitations carry over to more general settings, it would mean that reliance on a chain of command or other coordinating devices become even more important to combine stability and variety.

Grand schemes are interesting metaphors, but large-scale implementation is an issue. Hence, natural experiments gain traction as sources of inspiration with limited rationality. In addition, the contribution of types and their adaptability to different settings may differ.

Therefore, a tension between benefits of variety and stability remains. One set of alternative solutions involves a society with a multitude of games adapted to benefit certain types. The other path is to allow divergence in most regards and settings, except in some subset of desires – such as ideological disposition and propensity to obey – or exclusive patterns of interaction. Such multitudes correspond more closely to intuitive notions of organisations, institutions, networks, culture and habits. To manage such multitudes, the ruler will be interested in predictions about the likely patterns of interaction arising from incentive design. The following sections begins such inquiry.

STATISTICAL PROPERTIES OF NE

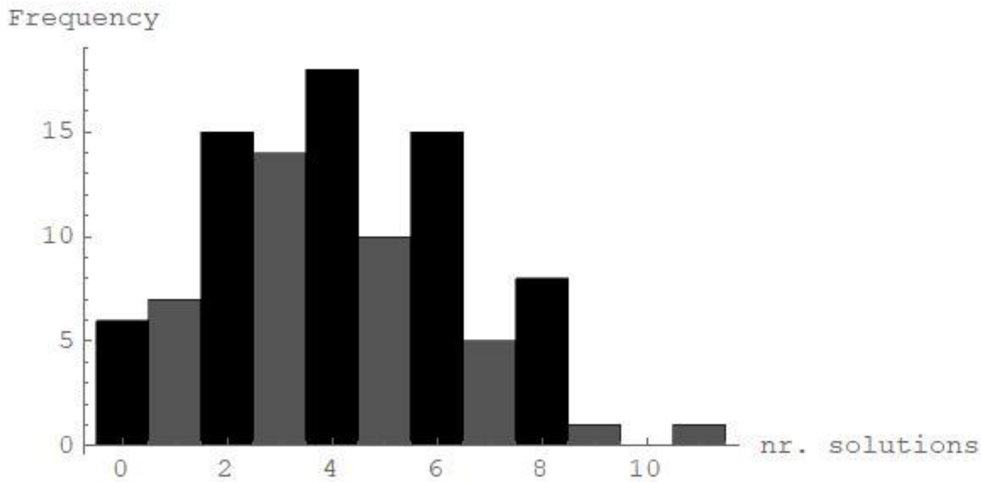
This section explores the relationship between the number of solutions and their type. 100 symmetric 2-player 50x50 games are generated with a Poisson**[3]** distribution for this purpose. Diagonal entries are drawn together.

T4. Distribution Solution Number

	Empty	**Odd**	**Even**
Frequency	6 %	38 %	56 %

6 % are only mixed-strategy equilibria, 38 % contain an odd number of solutions, and 56 % an even number.

F4. Nr. Solutions



In view of the discussion so far, coupled solutions should be most abundant. However, complement of the union of empty and symmetric solutions is not the coupled. For instance, by the aforementioned rationale, it is possible that a diagonal/symmetric solution offsets one of the coupled NE but not both. This however, should be relatively uncommon as there must be a coincidence of symmetric NE and coupled equilibria at either one column or row.

Hypothesis. Coupled solutions are most common, and the *number of solutions* is connected to the *solution type* in such way that odd numbers and symmetric go together, and even numbers favour coupled types.

The first part of the hypothesis is due to the probability-interference argument above. The second considers a relationship between number of solutions and their character/type. Odd numbers arise because there is only an odd number of symmetric; or there is a combination of symmetric and odd. If the distribution is such that symmetric are not a priori favoured, then the former is less common for numbers>1. More than one equilibrium should be favoured in bigger games, data in F4 show exactly that. Moreover, separated couples are uncommon. Hence, symmetric should be more common among odd-numbered solutions than in even-numbered.

By the same token, couples come in pairs, and thus favoured in even numbers in view of the characterisation of the (*symmetric solution*)–(*odd-number*) link.

T5. Distribution Solution Type

	Empty	**Coupled***	**Symmetric**
Frequency	1.5 %	61 %	37.6 %

Table 5 shows that the first part of the hypothesis cannot be rejected. Coupled are the most common solution-type. Coupled are 66 % more than symmetric.

The second part of the hypothesis cannot be rejected either.

T6. Distribution Number-Type Link

	Symmetric	**Coupled**
Even	33.6 %	66.4 %
Odd	44.2 %	55.8 %
Total	37.6 %	62.4 %

T6 shows coupled solutions are most common overall, in both subgroups. As hypothesised, symmetric are relatively less common in even than in odd-numbered solutions. In short, evidence suggests that the conditional probability of a symmetric solution given even-numbered solutions is less than the corresponding conditional when the number of solutions is odd.

PROBABILITY MODEL OF NE

The wise ruler has capacity beyond empirical studies, as those in the previous section, by conducting thought experiments to predict relevant distributions resulting from a society of games. Governance through adjustments of distributions will appear more natural than rule by direct decree. Adjustments of incentive distributions, combined with means to ensure a degree of robustness of the parameters and known reactions to given incentives, ensures predictions of the distribution of types.

For purpose of exposition, now consider a standard 2x2 game, with completely independently distributed incentive matrix. The distribution of the number of solutions in 1000 games generated with a Poisson**[3]** is given in F5. Data for this provided but figures in text refer to 100x1000.

By virtue of the definition of a NE and its implications, it is possible to make the following statistical model. The probability of 1 or 2 NE is

$P(1NE \vee 2NE) = 1 - \mathbf{[}P(4NE) + P(3NE) + P(|\emptyset|)\mathbf{]} = 90.4226\ \%$

The probability of four NE is

$P(4NE) = P\mathbf{[}(\Pi a1 = \Pi a3) \& (\Pi a2 = \Pi a4)\mathbf{]}\ P\mathbf{[}(\Pi b1 = \Pi b3) \& (\Pi b2 = \Pi b4)\mathbf{]} =$ */independence & equally distributed/* $= \mathbf{[}\boldsymbol{\Sigma}\, P(\Pi=i)^2\mathbf{]}^4 = 0.000771434$

$P(3\ NE) = 0.0347181 =$

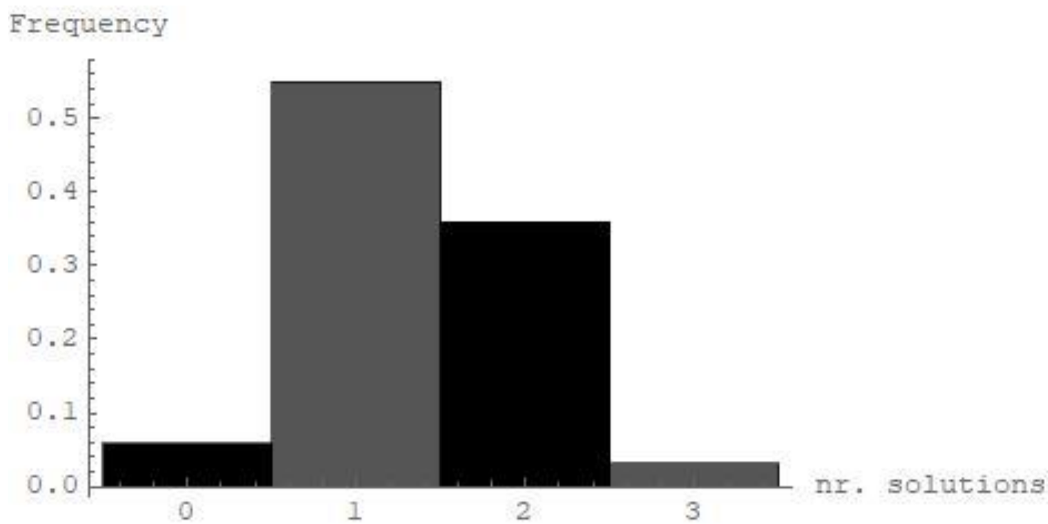
$$\binom{4}{3}\Big[\boldsymbol{\Sigma}\, P(\Pi=i)^2\ \boldsymbol{\Sigma}\big[P(\Pi>i)P(\Pi=i)\big]\Big]^2 + 2\big[\boldsymbol{\Sigma}\, P(\Pi=i)^2\big]^3\ \boldsymbol{\Sigma}\big[P(\Pi>i)P(\Pi=i)\big]\Big]$$

$P(|\emptyset|)] = 2\mathbf{[}\boldsymbol{\Sigma}\mathbf{[}P(\Pi>i)P(\Pi=i)\mathbf{]]}^4 = 0.0602843$

Player 1 is indexed with a, and player 2 with b. Indexation starts from m11 on the leftmost upper corner and proceeds row-wise. These figures are close to the empirical distribution with 90.35 % chance of getting 1 or 2 NE . Needless to say, the other are close too. Likewise, it is easy to generalise in terms of distributions, and the same should be the case for a greater number of strategies. However, more players *and* strategies may be more challenging.

Repeated samples show that the model is reasonable. Four NE are realised a few times every 10 samples, in line with the model. Future updates of this Footnote will likely expand more on modelling and statistical inference.

F5. Nr. Solutions



One can deduce an exact formula for the probability of symmetric and other equilibria in the example above. Larger games can also be analysed by similar means. Manipulation of the distribution for some players to e.g. induce a preferred mix of types is also feasible.

CONCLUDING REMARKS 3

Theoretical ruminations confined in a world governed by **g[]** motivated experiments on the diagonal of static games, and replicator dynamics. Although the former can sustain multiple types as NE these are not in general preserved in myopic dynamics, which favour homogeneity. Not even much higher and equal diagonal payoffs is enough.

This puts more requirements on institutions – e.g. chain of command or other coordinating devices, in order to attain both versatility and stability. Mixed-strategy NE can alter this conclusion.

The connection between the number of solutions and their character in terms of symmetric/diagonal or coupled solutions begins with an estimate of the externality of a solution's character on the conditional probability of other NE. The other part is the empirical distribution of purely mixed, odd- or even-numbered solutions, which also is natural to conjecture. The hypothesis is that coupled (off-diagonal) solutions in a symmetric matrix are most widespread; an even-number of solutions favour coupled ones; while an odd number favours symmetric solutions (diagonal) relatively more than an even number. Data could not reject the hypothesis, but it remains to be tested with relaxed assumptions.

Accurate predictions can be made by means of statistical modelling. In particular, it is possible to deduce an exact formula for the distribution of the number of solutions. It is also possible to deduce the distribution of symmetric and off-diagonal solutions. Such formulas do not rely on a particular class of tensors. Data lends support to the statistical model.

The rationale of this approach should be clear by now. Theoretical work is greatly aided by universal solvers as one can check the plausibility of results for the sake of logical intuition, hypothesis testing or creative theory building.

Please note: *This paper is a first draft, revision may result in major updates*. Comments, requests or questions are welcome.

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